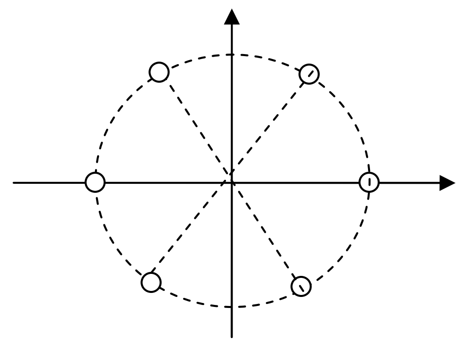
Math 656 • Midterm Examination • March 27, 2015 • Prof. Victor Matveev

1) (14pts) Find **all values** of z in polar or Cartesian form, and plot them as points in the complex plane:

(a) Since *n*-th root has exactly *n* values, there will be exactly 2×3=6 values, lying in the corners of a regular hexagon, with vertices on a circle of radius $\sqrt{2}$:

$$z=\frac{\left(1+\sqrt{3}i\right)^{1/2}}{i^{1/3}}=\frac{\left(2e^{i\left(\frac{\pi}{3}+2\pi j\right)}\right)^{1/2}}{\left(e^{i\left(\frac{\pi}{2}+2\pi k\right)}\right)^{1/3}}=\frac{\sqrt{2}e^{i\left(\frac{\pi}{6}+\pi j\right)}}{e^{i\left(\frac{\pi}{6}+\frac{2\pi}{3}k\right)}}=\sqrt{2}e^{i\pi\left(j-\frac{2k}{3}\right)}=\boxed{\sqrt{2}e^{\frac{i\pi n}{3}}}$$

$$n=0,\ \pm1,\ \pm2,\ 3$$



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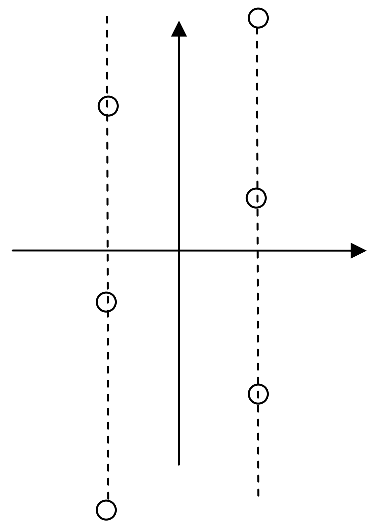
(b) $z=\cosh^{-1}(i)$ Infinitely many values, since cosh(z) has period of 2π*i*:

$$i=\cosh z=\frac{e^{z}+e^{-z}}{2}\ \Rightarrow\ e^{z}+e^{-z}=2i\ \Big\|\times\left(w=e^{z}\right)$$

$$\Rightarrow\ w^{2}-2iw+1=0\ \Rightarrow w=\frac{2i+(-4-4)^{1/2}}{2}=i\left(1\pm\sqrt{2}\right)$$

$$\Rightarrow\ z=\log w=\log i+\log\left(1\pm\sqrt{2}\right)=\boxed{\pm\ln\left(\sqrt{2}+1\right)+i\left(\pm\frac{\pi}{2}+2\pi k\right)}$$

Where we used $\log i=\frac{i\pi}{2}+i2\pi k$ and $\log\left(1-\sqrt{2}\right)=\underbrace{\ln\left(\sqrt{2}-1\right)}_{-\ln\left(\sqrt{2}+1\right)}-i\pi+i2\pi k$



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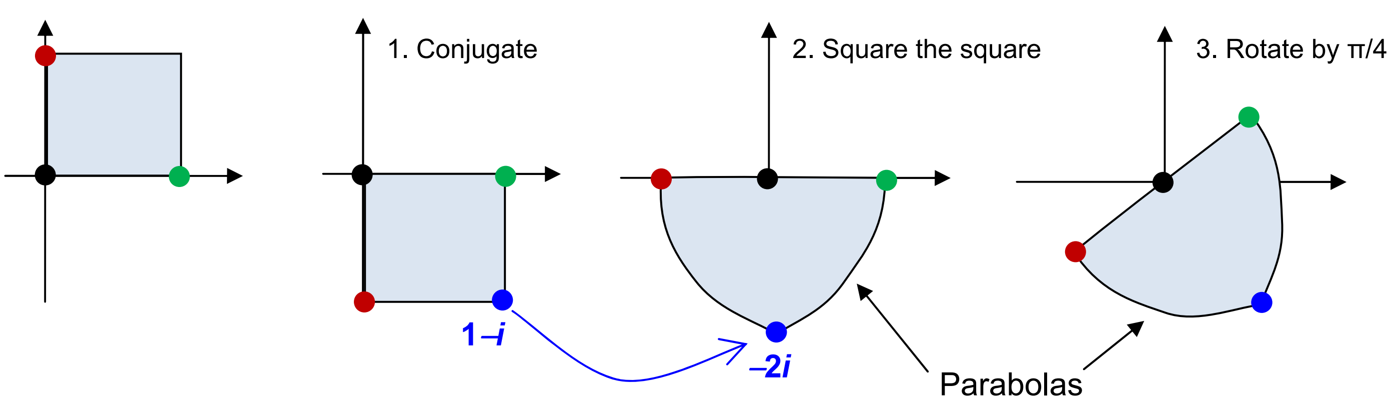
2) (13pts) Sketch the image of a square defined by vertices *z*=*i*, *z*=0, *z*=1 and z=1+*i* under the mapping $w=\frac{1+i}{\sqrt{2}}\left(\bar{z}\right)^{2}$. Hint: treat this mapping as a sequence of 3 simple transformations.



Step 1: Conjugation (flip around the Re(z) axis)

Step 2: Square: lines along axes remain lines, two other lines become parabolas:

$$\bar{z} = 1 + i\,y \;\Rightarrow\; (\bar{z})^2 = \underbrace{1 - y^2}_{u} + i\,\underbrace{(2y)}_{v} \;\Rightarrow\; u = 1 - \left(\frac{v}{2}\right)^2 \;\Leftarrow\; \text{A parabola}$$

$$\bar{z} = x - i \;\Rightarrow\; (\bar{z})^2 = \underbrace{x^2 - 1}_{u} + i\,\underbrace{(-2x)}_{v} \;\Rightarrow\; u = \left(\frac{v}{2}\right)^2 - 1 \;\Leftarrow\; \text{A parabola}$$

Step 3: Rotation by π/4: $w = \dfrac{1+i}{\sqrt{2}}(\bar{z})^2 = \exp\left(i\dfrac{\pi}{4}\right)(\bar{z})^2$

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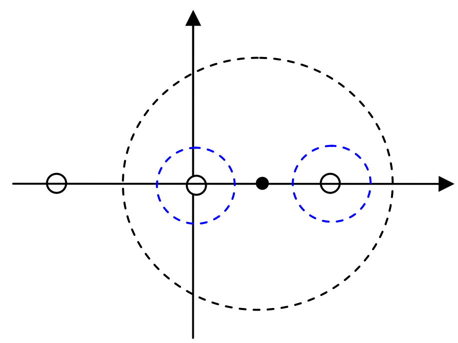
3) (21pts) Use an appropriate method to calculate each integral over the indicated contour

(a) Only two singularities are within the circle, so we can deform the contour to convert it to two integrals around each singularity, and then use the Cauchy Integral Formula for each:



$$\oint_{|z-1|=2} \frac{\cos z\; dz}{z^2\left(z^2-4\right)} = \oint_{|z|=\varepsilon} \frac{1}{z^2}\underbrace{\frac{\cos z}{z^2-4}}_{f(z)}\,dz + \oint_{|z-2|=\varepsilon} \frac{1}{z-2}\underbrace{\frac{\cos z}{z^2(z+2)}}_{g(z)}\,dz$$

$$= 2\pi i\, f'(0) + 2\pi i\, g(2)$$

$$= 2\pi i\underbrace{\left.\frac{-\sin z\left(z^2-2\right) - 2z\cos z}{\left(z^2-4\right)^2}\right|_{z=0}}_{0} + 2\pi i\frac{\cos 2}{16} = \boxed{\frac{\pi i \cos 2}{8}}$$

(b) $\displaystyle\oint_{|z|=1} \frac{z\,dz}{\left(e^z-1\right)^2}$ Integral over a circle of radius 1 (hint: find a couple dominant terms in the Laurent series)

$$\frac{z}{\left(e^z-1\right)^2} = \frac{z}{\left(z + \frac{z^2}{2} + O(z^3)\right)^2} = \frac{z}{z^2\left(1 + \frac{z}{2} + O(z^2)\right)^2} = \frac{1}{z}\frac{1}{1 + z + O(z^2)} = \frac{1}{z}\left(1 - z + O(z^2)\right) = \frac{1}{z} - 1 + O(z)$$

$$\Rightarrow \oint_{|z|=1} \frac{z\,dz}{\left(e^z-1\right)^2} = \oint_{|z|=1}\left(\frac{1}{z} - 1 + O(z)\right)dz = \boxed{2\pi i}$$

(c) $\int\limits_C \cosh\left(\log_\pi \bar{z}\right) dz$ C = semi-circle in the right half-plane of radius 1 centered at the origin and connecting point –*i* to point +*i* . Integrand is not analytic anywhere, so all we can do is parametrize:

$$\int\limits_C \cosh\left(\log_\pi \bar{z}\right) dz = \int\limits_{-\pi/2}^{\pi/2} \cosh\left(\log_\pi e^{-i\theta}\right) d\left(e^{i\theta}\right) = i\int\limits_{-\pi/2}^{\pi/2} \cosh\left(-i\theta\right) e^{i\theta} d\theta$$

$$= \frac{i}{2}\int\limits_{-\pi/2}^{\pi/2} \left(e^{+i\theta} + e^{-i\theta}\right) e^{i\theta} d\theta = \frac{i}{2}\int\limits_{-\pi/2}^{\pi/2} \left(e^{2i\theta} + 1\right) d\theta = \frac{i}{2}\left[\frac{e^{2i\theta}}{2i} + \theta\right]_{-\pi/2}^{\pi/2} = \boxed{\frac{i\pi}{2}}$$

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4) **(13pts)** Find an upper bound for $\left|\int\limits_C \frac{e^{iz}\log_o z\, dz}{z^2+4}\right|$, where the integration contour *C* is a straight line connecting point *z*=*i* to point *z*=1 (assume $0 \le \arg_o z < 2\pi$).

1. $|e^{iz}| = e^{-y} \le 1$ since $y \in [0,\ 1]$

2. $|\log_o z| = \sqrt{(\ln r)^2 + \theta^2} \le \sqrt{\left(\ln\frac{1}{\sqrt{2}}\right)^2 + \left(\frac{\pi}{2}\right)^2} = \frac{\sqrt{(\ln 2)^2 + \pi^2}}{2}$

   Since $|z| \in [1/\sqrt{2}, 1]$ and $\arg z \in [0, \pi/2]$

   Note: a stronger bound $|\log z| \le \frac{\pi}{2}$ is also correct, but needs proof

3. $|z^2+4| \ge |4 - |z^2|| \ge 4 - 1 = 3 \Rightarrow \frac{1}{|z^2+4|} \le \frac{1}{3}$

4. Length of contour: $L = \sqrt{2}$

$$\Rightarrow \left|\int\limits_C \frac{e^{iz}\log_o z\, dz}{z^2+4}\right| \le \boxed{\frac{1}{3}\sqrt{\frac{(\ln 2)^2 + \pi^2}{2}}}$$

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5) **(13pts)** Find the first three dominant terms in the Taylor series for function $f(z) = \frac{1}{1+\log_\pi z}$ near point *z*=1; indicate where the full series would converge.

$$f(z) = \frac{1}{1+\log_\pi z} = \frac{1}{1+\log_\pi(1+\zeta)} = \frac{1}{1+\underbrace{\zeta - \frac{\zeta^2}{2} + O(\zeta^3)}_{u}} = 1 - u + u^2 + O(u^3)$$

$$1 - \left(\zeta - \frac{\zeta^2}{2} + O(\zeta^3)\right) + \left(\zeta + O(\zeta^3)\right)^2 + O(\zeta^3) = 1 - \zeta + \frac{3\zeta^2}{2} + O(\zeta^3) = \boxed{1 - (z-1) + \frac{3(z-1)^2}{2} + O((z-1)^3)}$$

Nearest singularity is a pole at log(z)=–1, corresponding to z=exp(-1), therefore the radius of convergence is $|\zeta| = |z-1| < 1 - e^{-1}$

====================== **Pick any two problems between 6, 7, 8** ======================

6) (13pts) Suppose a given Laurent series $\sum_{k=-\infty}^{+\infty} c_k\left(z-z_o\right)^k$ has a maximal domain of convergence described by $0<r<|z-z_o|<R$. Does the principle part of this series converge anywhere outside this ring? What about the positive-power part, $\sum_{k=\mathbf{0}}^{+\infty} c_k\left(z-z_o\right)^k$?

Any Taylor series converges on some disk (not a ring), therefore:

1. The positive-power part is a Taylor series and therefore converges in a disk $|z-z_o|<R$
2. The principal part is a Taylor series in $\zeta=\dfrac{1}{z-z_o}$ (e.g., Taylor series around $z-z_o=\infty$) and therefore also has to converge in a disk $|\zeta|=\dfrac{1}{|z-z_o|}<\dfrac{1}{r}$, corresponding to $|z-z_o|=\dfrac{1}{|\zeta|}>r$

Note that the intersection of these two domains gives the ring $r<|z-z_o|<R$, explaining why Laurent series converges in a ring.

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7) (13pts) Find ***all*** series representations centered at $z_o=i$ for function $f(z)=\dfrac{1}{(z-i)^2(z+1)}$, and indicate their respective domains of convergence. Note: partial fractions are not needed in this problem

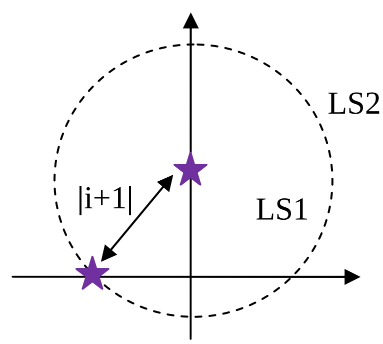
First, let's shift the argument: $\zeta\equiv z-i \;\Rightarrow\; f(z)=\dfrac{1}{(z-i)^2(z+1)}=\dfrac{1}{\zeta^2\left[\zeta+(1+i)\right]}$

Laurent series 1 ("local" Laurent series): factor out (1+i)

$$|\zeta|<|1+i| \;\Rightarrow\; \left|\frac{\zeta}{1+i}\right|<1 \;\Rightarrow$$

$$\frac{1}{\zeta^2(\zeta+1+i)}=\underbrace{\frac{1}{1+i}}_{\frac{1-i}{2}}\frac{1}{\zeta^2\left(1+\dfrac{\zeta}{1+i}\right)}=\frac{1-i}{2}\,\frac{1}{\zeta^2}\sum_{k=0}^{\infty}\left(\frac{-\zeta}{1+i}\right)^k$$

$$=\frac{1-i}{2}\,\frac{1}{\zeta^2}\sum_{k=0}^{\infty}\left(\frac{i-1}{2}\right)^k\zeta^k=\boxed{-\sum_{k=-2}^{\infty}\left(\frac{i-1}{2}\right)^{k+3}\zeta^k}$$



Laurent series 2: factor out ζ :

$$|\zeta|>|1+i| \;\Rightarrow\; \left|\frac{1+i}{\zeta}\right|<1 \;\Rightarrow\; \frac{1}{\zeta^2(\zeta+1+i)}=\frac{1}{\zeta^3\left(1+\dfrac{1+i}{\zeta}\right)}=\frac{1}{\zeta^3}\sum_{k=0}^{\infty}\left(-\frac{1+i}{\zeta}\right)^k=\boxed{\sum_{k=3}^{\infty}\frac{(-1-i)^{k-3}}{\zeta^k}}$$

8) (13pts) Make a **rough sketch** of the domain of convergence of the series $\sum_{k=0}^{+\infty} \frac{(\log_\pi z)^k}{k^2}$

Ratio test yields the condition $|\log_\pi z| < 1$

$$|\log_\pi z| < 1 \;\Rightarrow\; \sqrt{(\ln r)^2 + \theta^2} < 1 \Rightarrow \begin{cases} |\ln r| < 1 \\ |\theta| < 1 \end{cases} \Rightarrow \begin{cases} -1 < \ln r < 1 \\ -1 < \theta < 1 \end{cases} \Rightarrow \begin{cases} e^{-1} < r < e \\ -1 < \theta < 1 \end{cases}$$

Therefore, this domain lies within a ring sector, touching its boundaries at {θ=0; r=$e^{\pm 1}$} and {r=1 (ln r=0); θ=±1}